An Analysis of the "Quarter-Wave" Technique of Reducing the Errors in UHF and Microwave Impedance Measurement

WILLIAM E. LITTLE, DOYLE A. ELLERBRUCH, MEMBER, IEEE, AND GLENN F. ENGEN

Abstract—An analysis is given of the "quarter-wave" impedance measurement technique. This technique, which finds its widest potential application in conjunction with standing wave machines, permits the approximate elimination of the error due to residual reflection or VSWR. If the other sources of error are small, the potential reduction in error is in the ratio $|t_{11}|/2|S_{11}|$, where S_{11} and t_{11} are the residual reflection coefficients of the standing wave machine and quarter-wavelength section, respectively.

Introduction

The "QUARTER-WAVE" technique [1]–[3] of impedance measurement is, in fact, a modification of existing methods, and derives its name from the use of a quarter-wavelength section of precision waveguide in the measurement procedure. It finds its widest application in conjunction with standing wave machines, but has potential application to the reflectometer and other impedance measurement techniques.

Most standing wave machines suffer from a common problem, that of providing a low reflection transition between the uniform section of line in which the probe moves and the reference plane at which the unknown impedance is connected. In the simplest case, this involves only a transition from slotted to unslotted outer conductor; in practice, a change in cross section and dielectric supports may also be involved.

The residual reflection of this transition represents an error, which becomes increasingly important in the measurement of small reflections and whose elimination is the object of the quarter-wave technique. In a few words, the technique consists of first measuring the unknown impedance in the usual way and of taking a second measurement with a quarter-wavelength section of line inserted between the unknown and measuring machine. Ideally, this shifts the phase of the unknown by 180° while leaving its magnitude unaltered. Comparison of these two measurement results permits the (approximate) elimination of the residual reflection, and consequent determination of the unknown with improved accuracy.

It is the purpose of this paper to give an evaluation of the technique, including a tabulation of the various approximations and other sources of error. The analysis also gives first-order corrections which may be applied when the "quarter-wavelength" section deviates from the ideal value by a known amount.

Manuscript received September 26, 1966; revised May 12, 1967.

W. E. Little is with the Space Disturbance Lab., Institute of Telecommunication Sciences and Aeronomy, Boulder, Colo. He was formerly with the National Bureau of Standards.

D. A. Ellerbruch and G. F. Engen are with the National Bureau of Standards, Boulder, Colo. 80302

Basic Theory

The basic theory which underlies the method may be understood by reference to Fig. 1. The impedance measuring device is represented by an ideal impedance meter followed by a transition section that is described by the scattering coefficients S_{11}, S_{12}, S_{22} (reciprocity is assumed). The problem is thus one of relating the reflection coefficient of the unknown Γ_a to the reflection coefficient Γ_1 which is measured by the impedance meter. These quantities are related in the following way

$$\Gamma_1 = S_{11} + \frac{S_{12}^2\Gamma_a}{1 - S_{22}\Gamma_a}. \qquad (1)$$

In practice, $|S_{12}|^2 \approx 1$, while $|S_{11}|$ and $|S_{22}|$ are both small. ($|S_{11}|$ is one measure of the discontinuity in the transition section[1]). Inspection of (1) shows that the S_{11} term represents a potential error of increasing importance as the magnitude of the reflection coefficient $|\Gamma_a|$ decreases. In principle, S_{11}, S_{12}, and S_{22} may be measured, thus permitting Γ_a to be computed from the measured quantity Γ_1. Methods for doing this have been described by Deschamps [4], Altschuler and Felsen [5], and others [6], [7].

The quarter-wave technique provides an alternate solution to this problem. It differs from other techniques in that no correction has to be computed and applied to the measured quantity, because the residual error caused by the coefficient S_{11} is eliminated from the measured quantity.

Assuming the value of Γ_1 has been measured as indicated in Fig. 1, the next step is to insert the quarter-wave section as shown in Fig. 2. (In the general analysis to follow, the quarter-wave section is assumed to have the scattering coefficients t_{11}, t_{12}, t_{22}; for the present, it is considered "ideal.") Addition of the quarter-wave section transforms Γ_a into $-\Gamma_a$, and the corresponding value measured by the impedance meter is

$$\Gamma_2 = S_{11} - \frac{S_{12}^2\Gamma_a}{1 + S_{22}\Gamma_a}. \qquad (2)$$

[1] $|S_{11}| \approx$ (residual VSWR − 1)/2.

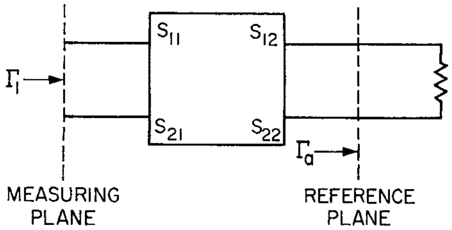

Fig. 1. Arrangement when reflection coefficient Γ_1 is measured.

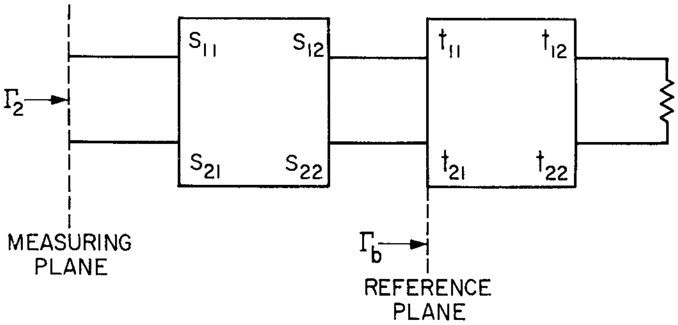

Fig. 2. Arrangement when reflection coefficient Γ_2 is measured.

Subtraction of (2) from (1) yields

$$\frac{\Gamma_1 - \Gamma_2}{2} = \frac{S_{12}^2}{1 - (S_{22}\Gamma_a)^2}\Gamma_a. \tag{3}$$

By hypothesis, $|S_{22}|$ is small, thus the second term in the denominator is of second order or less. The difference between $|S_{12}|^2$ and unity is closely related to the attenuation of the transition section and is no more than a few percent in typical cases. The elimination of the S_{11} term thus represents a substantial improvement in the potential accuracy, particularly for small values of $|\Gamma_a|$. In many instances, only the magnitude of Γ_a is of interest, and this is equal (approximately) to the magnitude of the left-hand side of (3).

Determination of the argument of Γ_a requires the argument of S_{12}^2. In complete analogy with conventional slotted line techniques, the unknown load may be replaced by a short circuit ($\Gamma_s = -1$). The observed reflection coefficient Γ_{1s} now becomes

$$\Gamma_{1s} = S_{11} - \frac{S_{12}^2}{1 + S_{22}}. \tag{4}$$

With the quarter-wave section inserted between the short circuit and measurement machine,

$$\Gamma_{2s} = S_{11} + \frac{S_{12}^2}{1 - S_{22}}. \tag{5}$$

Equations (4) and (5) may be combined to yield

$$\frac{\Gamma_{2s} - \Gamma_{1s}}{2} = \frac{S_{12}^2}{1 - S_{22}^2}, \tag{6}$$

which, in theory, represents a good approximation to S_{12}^2 in both phase and magnitude.

In practice, however, because of the problems and the errors associated with the determination of the magnitude of a large reflection, it is common to observe only the arguments (position of null) of Γ_{1s}, Γ_{2s}, and take the magnitudes as equal to unity.

Now,

$$\begin{aligned}\tfrac{1}{2}(\arg\Gamma_{1s} + \arg\Gamma_{2s} - \pi) &= \tfrac{1}{2}\arg(-\Gamma_{1s}\Gamma_{2s}) \\ &= \tfrac{1}{2}\arg\left\{\frac{S_{12}^4}{1 - S_{22}^2}\left[1 - \frac{2S_{11}S_{22}}{S_{12}^2} - \frac{S_{11}^2(1 - S_{22}^2)}{S_{12}^4}\right]\right\},\end{aligned} \tag{7}$$

and by inspection (ignoring second-order terms in S_{11}, S_{22}) this is (approximately) equal to $\arg(S_{12}^2)$.

In summary, if the second-order terms are ignored,

$$|\Gamma_a| \approx \left|\frac{\Gamma_1 - \Gamma_2}{2}\right|, \tag{8}$$

and

$$\arg\Gamma_a = \arg(\Gamma_1 - \Gamma_2) - \tfrac{1}{2}(\arg\Gamma_{1s} + \arg\Gamma_{2s} - \pi). \tag{9}$$

Equations (8) and (9) thus give the magnitude and argument of Γ_a in terms of the observed quantities Γ_1, Γ_2, $\arg(\Gamma_{1s})$, and $\arg(\Gamma_{2s})$.

By inspection it may be recognized that $\arg\Gamma_a$, as determined by (9), is invariant to the choice of phase reference in the slotted line (as of course it should be).

In the event that the electrical length βl of the quarter-wave section deviates from the prescribed value $\pi/2$ by a known amount, (8) becomes

$$|\Gamma_a| \approx \frac{|\Gamma_1 - \Gamma_2|}{2\cos\theta}, \tag{10}$$

where $\theta = \beta l - \pi/2$. For practical purposes, the technique for determining $\arg\Gamma_a$ remains the same, although an increased uncertainty results. These statements will be demonstrated in the section to follow.

Error Analysis

As noted above, the transition section is completely characterized by the scattering coefficients S_{11}, S_{12}, S_{22}, and since these are, in general, complex, they represent six parameters. However, if the transition is assumed lossless, and free of electrical discontinuities (matched), the following conditions are satisfied: $S_{11} = S_{22} = 0$, $|S_{12}| = 1$. The lossless and matched conditions are thus sufficient to determine five of these six parameters, leaving only the argument of S_{12} to be measured (by noting the position of the null when the transition is terminated by a short circuit). This, of course, is the basis for the conventional use of the slotted line.

A convenient measure of the transition discontinuity is provided by $|S_{11}|$, since this is (by definition) the reflection coefficient magnitude observed when the impedance meter (standing wave machine) is terminated by a reflectionless load.

The efficiency η_t, under this same condition (reflectionless load), is given by

$$\eta_t = \frac{|S_{12}|^2}{1 - |S_{11}|^2}. \tag{11}$$

In practice, $|S_{11}|$ is small, thus, $\eta_t \approx |S_{12}|^2$ and

$$1 - \eta_t \approx 1 - |S_{12}|^2 \tag{12}$$

will be taken as a measure of the adaptor loss. Finally, it may be noted that if the losses are small, $|S_{22}| \approx |S_{11}|$.

The foregoing comments also apply to the quarter-wavelength section with the exception that the phase shift is now usually inferred from physical measurements of its dimensions. Its departure from the ideal will be expressed in terms of its reflection, $|t_{11}|$, ($|t_{11}| \approx |t_{22}|$), efficiency η_q, and uncertainty in electrical wavelength, ε.

The object of the error analysis is thus to obtain an error expression for the above described technique that is correct to the first order in $|S_{11}|$, (or $|S_{22}|$), $(1-\eta_t)$, $|t_{11}|$, $(1-\eta_q)$, and ε. In practice, there will be additional errors in the measurements of Γ_1, Γ_2, arg Γ_{1s}, and arg Γ_{2s}, but the treatment of these is outside the scope of this paper. The purpose here is primarily one of comparing the quarter-wave technique with the more conventional procedure, and thus such things as probe coupling errors, etc., which are common to both procedures, are not considered.

In the more general treatment, (1) is unaltered, while (2) is replaced by

$$\Gamma_2 = S_{11} + \frac{S_{12}^2 \Gamma_b}{1 - S_{22}\Gamma_b}, \tag{13}$$

where

$$\Gamma_b = t_{11} + \frac{t_{12}^2 \Gamma_a}{1 - t_{22}\Gamma_a} \tag{14}$$

and the $t_{m,n}$ are the scattering coefficients of the quarter-wavelength section.

Equation (13) may be combined with (1) to yield

$$\begin{aligned}\Gamma_1 - \Gamma_2 &= \frac{S_{12}^2(\Gamma_a - \Gamma_b)}{(1 - S_{22}\Gamma_a)(1 - S_{22}\Gamma_b)} \\ &\approx S_{12}^2(\Gamma_a - \Gamma_b)[1 + S_{22}(\Gamma_a + \Gamma_b)],\end{aligned} \tag{15}$$

where terms in S_{22}^2 have been ignored.

By hypothesis, $|t_{11}|$ and $|t_{22}|$ are small and (14) may be written

$$\Gamma_b \approx t_{12}^2 \Gamma_a + t_{11} + t_{12}^2 t_{22} \Gamma_a^2, \tag{16}$$

which is correct to first order in t_{11} and t_{22}. Substitution of (16) into (15) yields[2]

$$\begin{aligned}\Gamma_1 - \Gamma_2 \approx S_{12}^2[&(1 - t_{12}^2)\Gamma_a - t_{11} - t_{12}^2 t_{22} \Gamma_a^2 \\ &+ S_{22}\Gamma_a^2(1 - t_{12}^4)].\end{aligned} \tag{17}$$

In accord with the previous discussion,

$$t_{12}^2 \approx \eta_q e^{j2(\beta l + \varepsilon)}, \tag{18}$$

where βl is the measured electrical length of the "quarter-wave" section, and ε represents an error in its determination. Now,

$$\beta l + \varepsilon = \pi/2 + \theta + \varepsilon. \tag{19}$$

To a first approximation, $1 - t_{12}^2$ is given by

$$\begin{aligned}1 - t_{12}^2 \approx 2e^{j\theta} \cos\theta \{1 + (1 + j\tan\theta) \\ \cdot [j\varepsilon - 1/2(1 - \eta_q)]\},\end{aligned} \tag{20}$$

while[3]

$$1 - t_{12}^4 \approx -4je^{j2\theta} \sin\theta \cos\theta. \tag{21}$$

Substitution of (20) and (21) into (17), and making use of the approximation

$$S_{12}^2 \approx \frac{S_{12}^2}{|S_{12}|^2}[1 - (1 - \eta_t)], \tag{22}$$

leads to

$$\begin{aligned}\frac{\Gamma_1 - \Gamma_2}{2\cos\theta} \approx \Gamma_a\left[\frac{S_{12}^2}{|S_{12}|^2} e^{j\theta}\right] \\ \cdot\Bigg\{1 - (1 - \eta_t) + (1 + j\tan\theta)[j\varepsilon - 1/2(1 - \eta_q)] \\ - 2jS_{22}\Gamma_a e^{j\theta}\sin\theta - \frac{t_{11} + t_{12}^2 t_{22}\Gamma_a^2}{2\Gamma_a e^{j\theta}\cos\theta}\Bigg\},\end{aligned} \tag{23}$$

and

$$\begin{aligned}\frac{|\Gamma_1 - \Gamma_2|}{2\cos\theta} \approx |\Gamma_a|\Bigg\{1 - (1 - \eta_t) - 1/2(1 - \eta_q) \pm \varepsilon\tan\theta \\ \pm 2|S_{22}|\,|\Gamma_a|\sin\theta \pm \frac{|t_{11}|(1 + |\Gamma_a|^2)}{2|\Gamma_a|\cos\theta}\Bigg\}.\end{aligned} \tag{24}$$

Here, the worst phase conditions have been assumed and the $\pm$ sign is used to indicate that the true value will lie somewhere within these limits.

In addition, two imaginary terms have been dropped, since they produce only a second-order change in the magnitude. Equation (24) is the desired generalization of (10) from which the following tabulation has been made.

TABLE I

FRACTIONAL ERRORS ($\Delta\Gamma_a/\Gamma_a$)

Source	Error
Dissipation in transition section	$-(1-\eta_t)$
Dissipation in quarter-wave section	$-1/2(1-\eta_q)$
Discontinuity in transition section	$\pm 2\|S_{22}\|\|\Gamma_a\| \sin\theta$
Discontinuity in quarter-wave section	$\pm\frac{\|t_{11}\|(1+\|\Gamma_a\|^2)}{2\|\Gamma_a\|\cos\theta}$
Uncertainty in length of quarter-wave section	$\pm\|\varepsilon\| \tan\theta$

[2] In this and what follows, it will be understood that the given equations are correct only to the first order in the quantities t_{11}, t_{22}, S_{22}, $(1-\eta_t)$, $(1-\eta_q)$, and ε.

[3] Note that only a "zero-order" approximation is required here, since $(1-t_{12}^4)$ is multiplied by S_{22} in (17).

The error due to dissipation in the transition section is the same as would occur in the conventional use of the slotted line, as could have been anticipated. The counterpart in the quarter-wave section is multiplied by one half, which may be explained by noting that this loss is present in only one of the two measurements. It is of interest to note that the errors due to both $|\varepsilon|$ and $|S_{22}|$ vanish (to first order) when θ is nominally zero (the section is within $\pm\varepsilon$ of being a quarter wavelength).

The most serious source of error is due to discontinuities in the quarter-wave section ($|t_{11}|$). By inspection these become of increasing importance as the value of the unknown reflection $|\Gamma_a|$ decreases. By way of comparison, the corresponding error in the conventional use of a slotted line is approximately $\pm|S_{22}|/|\Gamma_a|$; thus, the potential reduction in error is in the ratio $|t_{11}|/2|S_{11}|$, provided the other sources of error are negligible.

It should be noted that the preceding error expressions break down if $\theta \to \pi/2$, but should be satisfactory in the range $0<|\theta|<\pi/4$.

The error in the argument of Γ_a as given by (9) is determined next. Using the same reasoning as employed in the preceding paragraphs, a first-order expression for $\Gamma_{1s}\Gamma_{2s}$ is given by

$$\Gamma_{1s}\Gamma_{2s} \approx -S_{12}^4|t_{12}|^2 e^{j2\theta} \cdot \left\{1 + 2j\varepsilon + \left(S_{22} + \frac{S_{11}}{S_{12}^2 t_{12}^2}\right)2je^{j\theta}\sin\theta - \left(t_{22} + \frac{t_{11}}{t_{12}^2}\right)\right\}. \quad (25)$$

From the inspection of (23)

$$\arg(\Gamma_1 - \Gamma_2) \approx \arg\Gamma_a + \arg S_{12}^2 + \theta + \varepsilon - \tfrac{1}{2}(1-\eta_q)\tan\theta \pm \left[2|S_{22}||\Gamma_a|\sin\theta + \frac{|t_{11}|(1+|\Gamma_a|^2)}{2|\Gamma_a|\cos\theta}\right], \quad (26)$$

and similarly from (25),

$$\tfrac{1}{2}\arg(-\Gamma_{1s}\Gamma_{2s}) \approx \arg S_{12}^2 + \theta + \varepsilon \pm [2|S_{22}|\sin\theta + |t_{11}|]. \quad (27)$$

Subtraction of (27) from (26) yields

$$\arg(\Gamma_1 - \Gamma_2) - \tfrac{1}{2}\arg(-\Gamma_{1s}\Gamma_{2s}) = \arg\Gamma_a - \tfrac{1}{2}(1-\eta_q)\tan\theta \pm \left\{2|S_{22}|(1+|\Gamma_a|)\sin\theta + |t_{11}|\left[1 + \frac{(1+|\Gamma_a|^2)}{2|\Gamma_a|\cos\theta}\right]\right\}. \quad (28)$$

The last three terms thus give the error (in radians) in the determination of arg (Γ_a), by means of (9).

Conclusion

The error in a standing wave machine, introduced by the transition between the uniform slotted section and the reference plane of the machine, can often be significantly reduced by use of the "quarter-wave" technique described above, because the residual error caused by S_{11} is eliminated. This technique differs from others, such as the "sliding load method," in that the residual error is eliminated from the measured quantity, rather than explicitly evaluated; hence, computed corrections are not necessary.

This technique is of general applicability but probably has its greatest significance for coaxial line systems. In addition to a significant improvement in the accuracy potential of a given machine, its existing measurement capability may also be extended, with little loss in accuracy, to other waveguide or transmission line size by use of a low loss adaptor. In addition to the adaptor, the only requirement is for a precision quarter-wavelength section in the desired waveguide size.

Acknowledgment

The authors thank L. B. Elwell for his many helpful comments and suggestions during the preparation of this paper.

References

[1] A. Alford and C. B. Watts, Jr., "A wide band coaxial hybrid," *IRE Internat'l Conv. Rec.*, pt. 1, pp. 171–179, March 1956.

[2] R. E. Spinney, "The measurement of small reflection coefficients of components with coaxial connectors," *Proc. IEE* (London), vol. 110, p. 396, February 1963.

[3] A. E. Sanderson, "Calibration techniques for one and two port devices using coaxial reference air lines as absolute impedance standards," presented at the Instrument Soc. of Am. Conf., October 1964.

[4] G. A. Deschamps, "Determination of reflection coefficients and insertion loss of a waveguide junction," *J. Appl. Phys.*, vol. 24, p. 1046, August 1953.

[5] H. M. Altschuler and L. B. Felsen, "Network methods in microwave measurements," *Proc. Symp. Modern Advances in Microwave Techniques*. Brooklyn, N. Y.: Polytechnic Press, 1955, pp. 271–307.

[6] H. M. Altschuler, L. B. Felsen, R. E. Hammond, H. Kurss, and A. A. Oliner, "Instruction manual on equivalent circuit measurements of waveguide structures," Microwave Research Institute, Polytechnic Inst. of Brooklyn, Brooklyn, N. Y., Preport R-284-52, PIB-223, Contract AF-19(122)-3, pp. 176–181.

[7] *Handbook of Microwave Measurements*, 3rd ed. Brooklyn, N. Y.: Polytechnic Press, ch. 4.